INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 20, 523–539 (1995)

RECENT ADVANCES TOWARDS AN EFFECTIVE VIRTUAL-PULSE (VIP) EXPLICIT TIME INTEGRAL METHODOLOGY FOR GENERAL MULTIDIMENSIONAL THERMAL ANALYSIS

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SUMMARY

Recent advances and progress towards the development of an effective Virtual-Pulse (VIP) explicit time integral methodology of computation for applicability to general multidimensional non-linear transient thermal analysis of structures and materials is overviewed. The computational methodology is derived from new and different perspectives and the theoretical basis as well as the practical applicability to multidimensional thermal analysis situations are detailed. The VIP methodology inherits improved accuracy and superior stability characteristics in comparison to the traditional approaches customarily employed by thermal analysts. With the notion of providing techniques for high-speed computing environments and parallel architectures, the present approach is developed for such computations and also ideally suited for personal workstation computing environments. Results of the numerical test models for multidimensional problems validate the overall concepts for general applicability to thermal analysis situations.

KEY WORDS: virtual pulse; thermal analysis

INTRODUCTION

The computational aspects of thermal heat transfer in engineering structures and materials is important since non-linear and/or non-uniform heating may have a significant effect on the performance characteristics and is of utmost concern in the development of advanced structural materials, component behaviour, and structural design. Several different techniques of numerical analysis of transient non-linear/linear thermal problems exist. Finite difference, finite element, finite volume, boundary element methods and the like are now quite familiar approaches for the space discretization. And, as such after the so-called semi-discretization process which leads to a system of first-order ordinary differential equations in time, numerous time discretization approaches exist and have been employed for evaluating the thermal response. This includes some form of finite difference approximations for the time derivatives which lead to direct time stepping methods; modal analysis which employ modal decomposition to decouple the governing

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CCC 0271–2091/95/060523–17

Received 18 March 1994
Revised 2 November 1994

equations, and then typically involve a step-by-step time integration procedure to approximate the temperature field synthesized from the modes; hybrid transfinite element formulations which involve transform methods in conjunction with the semi-discrete equations and numerical inversion to yield temperature solutions at desired times of interest; approximations in space and time and the like.[1-9]

With the advent of modern computing technology, numerous solution algorithms and strategies have been proposed for handling general non-linear transient thermal analysis problems. Of most of the existing methods, direct time stepping methods have been the most popular and widely advocated in most commercial codes. The so-called trapezoidal family of α-methods[2] have indeed played a dominant role and well received both by the research community at large and code developers. Of these, only the implicit Crank–Nicolson method[10] is second-order time accurate and unconditionally stable. Various other time stepping formulations have also been proposed for heat transfer problems to include the multi-step Lee's algorithm,[11] and those proposed by Zlamal[12] and Liniger.[13] Investigations employing related implicit and explicit solution techniques for thermal analysis of structures appear due to Adelman *et al.*[14] More recent efforts due to Tamma and Namburu[15] describe developments in explicit approaches which maintain second-order accuracy; however, they are only conditionally stable. Unconditionally stable explicit methods have also been proposed such as the DuFort–Frankel scheme[16] and Saul'yev's method;[17] however, they suffer from the drawbacks of conditional consistency which implies convergence can be achieved if the time step Δt approaches zero much more rapidly than the mesh size $h \to 0$. The time step restriction is much severe for a desired accuracy in comparison to the unconditionally stable implicit Crank–Nicolson method. An extension of the Saul'yev's method in combination with finite elements for multidimentional problems appears by Trujillo.[18]

In comparison to direct time integration methods, modal analysis methods indeed exist in certain codes and have been mostly applied to linear problems and for long transient durations. However, their drawbacks stem from the fact that for transient non-linear analysis, there is a definite lack of effective and efficient use of such methods. In particular from the context of analogous dynamic structural problems, the need to recalculate eigenproblems frequently enough so that local mode superposition is permissible during a typical non-linear transient analysis, makes such approaches extremely cumbersome and inefficient. As a consequence, these methods have not been taken seriously for transient non-linear thermal problems.

Unlike past approaches for general heat transfer computations, the present paper describes new and recent advances and trends towards providing an effective and accurate Virtual-Pulse (VIP) time integral method of computation. With the advent of high-speed computing technology and the importance of parallel computations for efficient use of computing architectures, a major motivation for the present developments is also to permit an explicit solution methodology with effective stability, accuracy and computational characteristics. Of subsequent interest are also thermal applications to personal workstation computing environments. The theoretical basis and the issues relevant to accuracy and stability with emphasis on simple single degree of freedom (SDOF) linear heat transfer models to illustrate merely the basic concepts of the VIP methodology is described by Chen. *et al.*[19] The theoretical basis, the formulation and the developments of the VIP methodology with applications to non-linear transient single degree of freedom models are introduced by Tamma *et al.*[20] and, extensions of the VIP methodology with applications to finite element computations appeared more recently due to Mei *et al.*[21] In the present paper, recent theoretical and computational advances in the development and applicability of an explicit VIP methodology of computation is described and numerical test cases of practical engineering importance relevant to multidimensional applications are presented to support the overall developments for general non-linear transient heat transfer problems.

GOVERNING THERMAL MODEL EQUATIONS

Of general interest are the class of transient non-linear/linear thermal problems governed by

$$\rho(T)c(T)\frac{\partial T}{\partial t} - (k_{ij}\,T_{,j})_{,i} = Q \quad \text{in } \Omega \tag{1a}$$

with appropriate boundary and initial conditions given by

$$T = T_{\mathrm{p}} \quad \text{on } S_1 \tag{1b}$$

$$\mathbf{q}\cdot\hat{\mathbf{n}} - q_s + q_h + q_r = 0 \quad \text{on } S_2 \tag{1c}$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the surface S_2 and pointing outward to this surface boundary.

$$q_h = h(T)(T - T_h) \tag{1d}$$

$$q_r = \sigma\varepsilon(T)(T^4 - T_r^4) \tag{1e}$$

where q_s is the surface heat flux, q_h is the convective heat flux, and q_r is the heat flux due to radiation acting on S_2 of the boundary. T is the temperature field, k_{ij} is the thermal conductivity tensor, ρ is the density, c is the specific heat and Q is the internal heat source per unit volume. On the boundary S_1, T_{p} is the prescribed temperature field. On boundary S_2, the natural boundary conditions involve h which is the convection heat transfer coefficient with T_h as the convective medium temperature, σ is the Stefan–Boltzmann constant, ε is the surface emissitivity and T_r is the radiation medium temperature. In general, the thermophysical properties (k, ρ, c, h, ε, etc.) are all temperature dependent.

The initial conditions may be assumed as

$$T(x_i, 0) = T_i \tag{1f}$$

Finite element formulations: space discretization

Introducing the standard finite element procedure, the semi-discrete non-linear transient thermal model equations are typically of the form represented by

$$\mathbf{C}(\mathbf{T}, t)\dot{\mathbf{T}} + \mathbf{K}(\mathbf{T})\mathbf{T} = \mathbf{Q}(t) \tag{2a}$$

$$\mathbf{T}(0) = \mathbf{T}_i \tag{2b}$$

where

$$\mathbf{C} = \sum \int_{\Omega^e} \rho c\, N_\alpha N_\beta \,\mathrm{d}\Omega \tag{3a}$$

$$\mathbf{K}_c(\mathbf{T}) = \sum \int_{\Omega^e} k_{ij}\, N_{\alpha,i} N_{\beta,j} \,\mathrm{d}\Omega \tag{3b}$$

$$\mathbf{K}_h(\mathbf{T}) = \sum \int_{S_2^e} h\, N_\alpha N_\beta \,\mathrm{d}S \tag{3c}$$

$$\mathbf{K}_r(\mathbf{T})\mathbf{T} = \sum \int_{S_2^e} \sigma\varepsilon\, T^4 N_\alpha \,\mathrm{d}S \tag{3d}$$

$$\mathbf{Q}_Q = \sum \int_{\Omega^e} Q\, N_\alpha \,\mathrm{d}\Omega \tag{3e}$$

$$\mathbf{Q}_{qs} = \sum \int_{S_2^e} q_s N_\alpha \, dS \tag{3f}$$

$$\mathbf{Q}_h = \sum \int_{S_2^e} h T_h N_\alpha \, dS \tag{3g}$$

$$\mathbf{Q}_r = \sum \int_{S_2^e} \sigma \varepsilon T_r^4 N_\alpha \, dS \tag{3h}$$

and $\sum$ indicates summation over all elements in the physical domain, and all the thermophysical properties may be in general, dependent upon temperature.

Time discretization: trapezoidal α-family

The commonly employed solution algorithms for solving equations (2) have traditionally been the trapezoidal α-family of one-step methods given by

$$\mathbf{C}^{n+\alpha}\dot{\mathbf{T}}^{n+\alpha} + \mathbf{K}\mathbf{T}^{n+\alpha} = \mathbf{Q}^{n+\alpha} \tag{4a}$$

$$\mathbf{T}^{n+1} = \mathbf{T}^n + \Delta t \dot{\mathbf{T}}^{n+\alpha} \tag{4b}$$

$$\mathbf{T}^{n+\alpha} = (1-\alpha)\mathbf{T}^n + \alpha \mathbf{T}^{n+1} \tag{4c}$$

where $\mathbf{T}^n$ and $\dot{\mathbf{T}}^n$ are the representation of $\mathbf{T}(t_n)$ and $\dot{\mathbf{T}}(t_n)$, respectively; $\mathbf{Q}^{n+\alpha} = \mathbf{Q}(t_{n+\alpha})$ and Δt is the time step. Of these, only $\alpha = 0{\cdot}5$ which is the trapezoidal rule enables the method to be implicit, second-order accurate and unconditionally stable. The explicit solution method for $\alpha = 0$ is only conditionally stable and first-order accurate.

VIRTUAL-PULSE (VIP) TIME INTEGRAL METHODOLOGY

Unlike past approaches, the so-called Virtual-Pulse (VIP) time integral methodology of computation is developed from new and different perspectives and a fundamentally sound theoretical basis for general applications encompassing non-linear/linear thermal heat transfer computations.

After the so-called semi-discretization process, the discretization in the time domain is achieved via a weighted residual process with the time weighting functions proposed via the present developments being uniquely selected so as to account for the physics involved, thereby, resulting in a general explicit time integral type methodology which possesses improved accuracy and stability characteristics (in comparison to the widely advocated traditional time stepping method), and has computationally attractive and effective features. The resultant weak form obtained via these developments appears in an integral form, and as a consequence, no finite difference approximations need to be involved. And, unlike direct time integration methods and mode superposition techniques, the proposed time integral methodology of computation capitalizes on the advantages of both via a modal transformation introduced to obtain primarily appropriate physically related weighting functions and then derives the resultant weak form from which an effective explicit computational methodology results.

Generalized formulations and developments

Time discretization: VIP methodology of computation. The general transient non-linear semi-discretized thermal formulations can be typically represented as

$$\mathbf{C}(\mathbf{T})\dot{\mathbf{T}} + \mathbf{F}(\mathbf{T}) = \mathbf{Q}(t) \tag{5a}$$

$$\mathbf{T}(0) = \mathbf{T}_i \tag{5b}$$

where $\mathbf{C} \in \mathbf{R}^N \times \mathbf{R}^N$; $\mathbf{F}$, $\mathbf{T}$, $\dot{\mathbf{T}}$ and $\mathbf{Q} \in \mathbf{R}^N$. N is the total number of degrees of freedom in the problem, $\mathbf{C}$ is the general temperature-dependent heat capacity matrix which is symmetric and positive-definite, $\mathbf{F}(\mathbf{T})$ is the representative conductivity matrix influenced by conduction/convection/radiation ($\mathbf{F}(\mathbf{T}) = \mathbf{K}_c(\mathbf{T})\mathbf{T} + \mathbf{K}_h(\mathbf{T})\mathbf{T} + \mathbf{K}_r(\mathbf{T})\mathbf{T}$), and $\mathbf{Q}$ is the load vector from the various contributions involving internal heat generation, surface heating, convection, radiation and the like ($\mathbf{Q} = \mathbf{Q}_Q + \mathbf{Q}_{qs} + \mathbf{Q}_h + \mathbf{Q}_r$).

The theoretical developments follow next. In the time interval $[t_n, t_{n+1}]$ consider an arbitary time moment t such that

$$t = t_n + \gamma(t_{n+1} - t_n), \quad 0 \leqslant \gamma \leqslant 1 \tag{6}$$

Merely for the purpose of describing the development of the VIP methodology, consider $\mathbf{C}$ to be comprised of $\mathbf{C}_L$ (linear part) and $\mathbf{C}_{NL}$ (non-linear part) as

$$\mathbf{C} = \mathbf{C}_L + \mathbf{C}_{NL} \tag{7}$$

Introducing this into equation (5), we have at any arbitrary γ,

$$\mathbf{C}_L \dot{\mathbf{T}}^\gamma + \mathbf{F}^\gamma = \mathbf{Q}^\gamma - \mathbf{C}_{NL}^\gamma \dot{\mathbf{T}}^\gamma = \mathbf{R}_{eq}^\gamma$$

The time discretization follows next via a unique weighted residual procedure, which, instead of employing standard polynomial functions in the time domain (which leads to finite difference approximations), seeks to incorporate the representative behaviour of the nature of the heat transfer problems as follows:

$$\int_{t_n}^{t_{n+1}} \delta\mathbf{U}^T(\mathbf{C}_L \dot{\mathbf{T}}^\gamma + \mathbf{F}^\gamma - R_{eq}^\gamma)dt = 0, \quad t_n, t_{n+1} \in [0, t_{end}] \tag{8}$$

where t_{end} is the time duration, $\delta\mathbf{U}^T$ are the appropriate weighting functions to be determined, and satisfy

$$\delta\mathbf{U}^T(t_n) = 0 \tag{9}$$

Integrating equation (8) by parts, we have

$$\delta\mathbf{U}^T\mathbf{C}_L\mathbf{T}^\gamma\Big|_{t_n}^{t_{n+1}} - \int_{t_n}^{t_{n+1}} \delta\dot{\mathbf{U}}^T\mathbf{C}_L\mathbf{T}^\gamma dt + \int_{t_n}^{t_{n+1}} \delta\mathbf{U}^T\mathbf{F}^\gamma dt - \int_{t_n}^{t_{n+1}} \delta\mathbf{U}^T\mathbf{R}_{eq}^\gamma dt = 0 \tag{10}$$

or equivalently

$$\delta\mathbf{U}^T\mathbf{C}_L\mathbf{T}^\gamma\Big|_{t_n}^{t_{n+1}} - \int_{t_n}^{t_{n+1}} (\delta\dot{\mathbf{U}}^T\mathbf{C}_L - \delta\mathbf{U}^T\mathbf{K}_L)\mathbf{T}^\gamma dt$$
$$- \int_{t_n}^{t_{n+1}} (\delta\mathbf{U}^T\mathbf{K}_L\mathbf{T}^\gamma - \delta\mathbf{U}^T\mathbf{F}^\gamma)dt - \int_{t_n}^{t_{n+1}} \delta\mathbf{U}^T\mathbf{R}_{eq}^\gamma dt = 0 \tag{11}$$

where $\mathbf{K}_L$ is the linear part of the conductance matrix.

Since the weighting functions are arbitary, we propose the choice of these for the time interval $[t_n, t_{n+1}]$ as those from the linear solution of

$$\mathbf{C}_L \delta\dot{\mathbf{U}} - \mathbf{K}_L \delta\mathbf{U} = \mathbf{C}_L \mathbf{X}\delta(t-\tau), \quad \tau \in [t_n, t_{n+1}] \tag{12}$$

where

$$\mathbf{X} = [\mathbf{X}_1, \mathbf{X}, \ldots, \mathbf{X}_m], \quad \mathbf{X} \in \mathbf{R}^N \times \mathbf{R}^M$$

$$\delta\mathbf{U} \in \mathbf{R}^N \times \mathbf{R}^M$$

and $\delta(t-\tau)$ is a Dirac delta function.

The solution of equation (12) is given by

$$\delta\mathbf{U} = \mathbf{X}\mathbf{Z}(t-\tau), \quad t \geqslant \tau \tag{13a}$$

$$\delta\mathbf{U} = 0, \quad t < \tau \tag{13b}$$

where

$$\mathbf{Z}(t-\tau) = \operatorname{diag}[e^{\lambda_i(t-\tau)}]$$

and λ_i, $i = 1, 2, \ldots, m$ are the eigenvalues of the linear problem given by

$$(\mathbf{K}_L - \lambda_i \mathbf{C}_L)\mathbf{X}_i = 0 \tag{14}$$

and $\mathbf{X}_i$ are the corresponding eigenvectors.

Substituting the proposed weighting functions, (13) into (11), at $\tau = t_n$ we have

$$\mathbf{f}(\mathbf{T}^{n+1}) = \mathbf{X}^T \mathbf{C}_L \mathbf{T}^n - \mathbf{Z}(t_{n+1}-t_n)\mathbf{X}^T \mathbf{C}_L \mathbf{T}^{n+1} + \int_{t_n}^{t_{n+1}} \mathbf{Z}(t-t_n)\mathbf{X}^T(\mathbf{K}_L \mathbf{T}^\gamma - \mathbf{F}^\gamma)dt$$

$$+ \int_{t_n}^{t_{n+1}} \mathbf{Z}(t-t_n)\mathbf{X}^T \mathbf{R}_{eq}^\gamma \, dt = 0 \tag{15}$$

or equivalently, introducing $\mathbf{T} = \mathbf{X}\mathbf{T}^*$ to transform from a finite element basis to the modal basis together with the relation

$$\mathbf{T}^\gamma = (\mathbf{1}-\gamma)\mathbf{T}^n + \gamma\mathbf{T}^{n+1} \tag{16}$$

we have the general non-linear relationship described by equations (15) represented as

$$\mathbf{f}(\mathbf{T}^{*n+1}) = 0 \tag{17}$$

MODIFIED NEWTON–RAPHSON PROCEDURE

A modified Newton–Raphson iteration procedure is adopted for the solution of equation (17) as follows:

$$\mathbf{J}_{m=0} \Delta\mathbf{T}_{m+1}^{*n+1} = -\mathbf{f}_m^{n+1} \tag{18a}$$

$$\mathbf{T}_{m+1}^{*n+1} = \mathbf{T}_m^{*n+1} + \Delta\mathbf{T}_{m+1}^{*n+1} \tag{18b}$$

and

$$\mathbf{T}^{n+1} = \mathbf{X}\mathbf{T}^{*n+1} \tag{18c}$$

An important feature of the present VIP methodology is the explicit nature of the resulting computational approach depicted by equation (18). The simplicity and effective nature of the

proposed time integral methodology for general transient non-linear problem lies in the actual computation of $\mathbf{J}_{m=0}$ and $\mathbf{f}_m^{n+1}$ which results in an explicit computational procedure involving iterations. These are detailed next.

Computation of Jacobian $\mathbf{J}_{m=0}$

The computational aspects associated with the evaluation of $\mathbf{J}_{m=0}$ are briefly outlined here. The Jacobian is obtained as

$$J_{ij} = \frac{\partial f_i}{\partial T_j^{*n+1}} \tag{19}$$

where $f_i(\mathbf{T}^{*n+1})$ is given by equation (17) [see also (15)].
Introducing the relations

$$\mathbf{T}^{\gamma} = (\mathbf{1} - \gamma)\mathbf{T}^n + \gamma\mathbf{T}^{n+1} \tag{20}$$

$$t = t_n + \gamma(t_{n+1} - t_n) \tag{21}$$

and

$$\mathbf{C}_{\mathrm{NL}}^{\gamma} \approx \mathbf{C}_{\mathrm{NL}}(\mathbf{T}^{\beta}) \tag{22}$$

where $\mathbf{T}^{\beta} = (\mathbf{1} - \beta)\mathbf{T}^n + \beta\mathbf{T}^{n+1}$ and β is a constant parameter within the range [0, 1] and is intended for improving the computational efficiency. It normally has the value of 0·5 based on numerical experiments.

The computation for $\mathbf{J}_{m=0}$ after algebraic manipulations simply results in a diagonal form given by

$$\mathbf{J}_{m=0} = \mathrm{diag}(-\mathrm{e}^{-\lambda_i \Delta t}\,\mathbf{C}_{\mathrm{L}ii}) \tag{23}$$

where $\mathbf{C}_{\mathrm{L}ii}$ are the contribution of the diagonal components of the diagonalized linear capacitance matrix, $\mathbf{X}^{\mathrm{T}}\mathbf{C}_{\mathrm{L}}\mathbf{X}$ and $\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m]$, $\mathbf{X} \in \mathbf{R}^n \times \mathbf{R}^m$.

Computation of $\mathbf{f}_m$

The computational aspects associated with the evaluation of $-\mathbf{f}_m$ follow analogous developments. Introducing the relations

$$\mathbf{T}^{\gamma} = (\mathbf{1} - \gamma)\mathbf{T}^n + \gamma\mathbf{T}^{n+1} \tag{24}$$

and

$$\mathbf{F}^{\gamma} = (\mathbf{1} - \gamma)\mathbf{F}^n + \gamma\mathbf{F}^{n+1} \tag{25}$$

to account accurately for the calculation of the representative terms, leads to the following representations after algebraic manipulations as

$$-\mathbf{f}_m^{n+1} = (\mathbf{Y}_1 - \mathbf{Y}_2)\mathbf{X}^{\mathrm{T}}(\mathbf{Q}^n - \mathbf{F}^n) + \mathbf{Y}_2\mathbf{X}^{\mathrm{T}}(\mathbf{Q}^{n+1} - \mathbf{F}_m^{n+1}) - \mathbf{Y}_3\mathbf{X}^{\mathrm{T}}\mathbf{C}(\mathbf{T}^{\beta})(\mathbf{T}_m^{n+1} - \mathbf{T}^n) \tag{26a}$$

where

$$\mathbf{Y}_1 = \mathrm{diag}\left[\frac{1}{\lambda_i}(\mathrm{e}^{\lambda_i \Delta t} - 1)\right] \tag{26b}$$

$$\mathbf{Y}_2 = \text{diag}\left\{\frac{1}{\lambda_i^2 \Delta t}\left[(\lambda_i \Delta t - 1)e^{\lambda_i \Delta t} + 1\right]\right\} \tag{26c}$$

$$\mathbf{Y}_3 = \text{diag}\left[\frac{1}{\lambda_i \Delta t}(e^{\lambda_i \Delta t} - 1)\right] \tag{26d}$$

Remarks

1. In the above VIP time integration methodology of computation for general transient non-linear problems, it is noteworthy to point out that the computation of the linear eigenproblem is done only once for the entire transient duration. Furthermore, instead of solving an N equation system, solution of m equations ($m < N$) will suffice; however, this may be problem dependent.
2. The explicit nature of the methodology makes it an attractive tool for applications to high-speed computing environments and parallel computations. The approach is also suitable for applications on personal workstation computing environments.
3. The proposed formulations yield improved accuracy and stability attributes in comparison to the generalized α-family of trapezoidal methods even for non-linear situations. Issues relevant to the notion of stability and accuracy from the context of linear situations are discussed subsequently.

Reduction to transient linear problems

It should be noted that for linear transient situations, equations (18) readily yield the solution response in one iteration. As a consequence, it can also be readily shown that for linear problems, the resulting formulations reduce to the explicit time stepping strategy (without any iterations) given by

$$\mathbf{T}^{*n+1} = \text{diag}(e^{-\lambda_i \Delta t})\mathbf{T}^{*n} + \text{diag}\left(\frac{e^{-\lambda_i \Delta t}}{C_{ii}}\right)\bar{\mathbf{R}} \tag{27a}$$

where

$$\bar{\mathbf{R}} = (\mathbf{Y}_1 - \mathbf{Y}_2)\mathbf{X}^{\mathrm{T}}\mathbf{Q}^n + \mathbf{Y}_2\mathbf{X}^{\mathrm{T}}\mathbf{Q}^{n+1} \tag{27b}$$

and, for constant thermophysical properties, all terms on the right-hand side of equation (27) are known.

The representative nodal temperatures can now be readily obtained employing

$$\mathbf{T}^{n+1} = \mathbf{X}\mathbf{T}^{*n+1} \tag{28}$$

Stability and accuracy characteristics

A brief overview of the stability and accuracy characteristics of the present VIP explicit methodology of computation is presented next as applied to linear transient heat transfer computations.

Following the standard procedure of stability analysis, we consider the SDOF (single degree of freedom) homogeneous modal equation

$$\dot{T} + \lambda T = 0 \tag{29}$$

whose analytical solution of the amplification factor is $e^{-\lambda \Delta t}$. For the explicit VIP time integral

methodology, the temporally discrete homogeneous modal equation can be readily obtained from equation (27) as

$$\mathbf{T}^{*n+1} = \mathrm{e}^{-\lambda \Delta t}\mathbf{T}^{*n} \tag{30}$$

Hence,

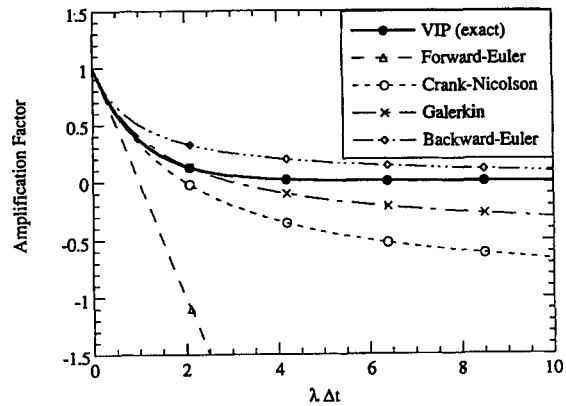
$$\text{Amplification factor } A = \mathrm{e}^{-\lambda \Delta t} \tag{31}$$

which is the same as the exact solution. For stability, the requirement is $|A| < 1$. Thus, it is obvious that the proposed explicit VIP time integral methodology is unconditionally stable. Comparative stability characteristics with the generalized trapezoidal α-family is depicted in Figure 1.

The temporally discrete SDOF modal problem for the representation (equation (27)) can be rewritten as

$$\mathbf{T}^{*n+1} = \mathbf{A}\mathbf{T}^{*n} + \mathbf{L}_n \tag{32}$$

where $\mathbf{L}_n$ is the heat loading operator at time $t = t_n$. An analysis of the local truncation error provides the accuracy with which the VIP methodology converges to the exact solution. Replacing T^{*n+1} and T^{*n} by the corresponding exact values, the local truncation error for the



Method	Type	Stability	Accuracy
VIP	Explicit	Unconditional	2*
Forward Euler	Explicit	Conditional	1
Crank-Nicolson	Implicit	Unconditional	2
Galerkin	Implicit	Unconditional	1
Backward Euler	Implicit	Unconditional	1

* VIP method provides an exact solution in the case of a linear heat load.

Figure 1. Comparative stability/accuracy of VIP versus trapezoidal family of methods

VIP methodology is governed by

$$|r(t_n)| \leqslant O(\Delta t^3) \tag{33}$$

for a general heat loading situation. Hence, for these general situations, the VIP methodology is second-order accurate and the method is consistent. In view of the previously mentioned considerations, the notion of convergence via the present developments is thus automatically established.

From the above-mentioned developments, the following remarks can be drawn for linear transient situations.

Remarks

1. The VIP methodology for linear problems is explicit (without any iterations) and unconditionally stable with second-order accuracy and consistent; the temperature field is readily obtained at any desired time of interest without loss of accuracy for linear variation in heat loads.
2. It is second-order accurate for an arbitrary heat load situation.
3. The capacitance matrix C does not have to be physically lumped since the use of modal co-ordinates and M-orthogonality automatically decouples the resulting formulations.
4. The methodology is directly self-starting.

NUMERICAL TEST EXAMPLES

Comparative numerical test models are evaluated in this section. All comparisons are purposely made with the implicit second-order accurate Crank–Nicolson method and the present sec-

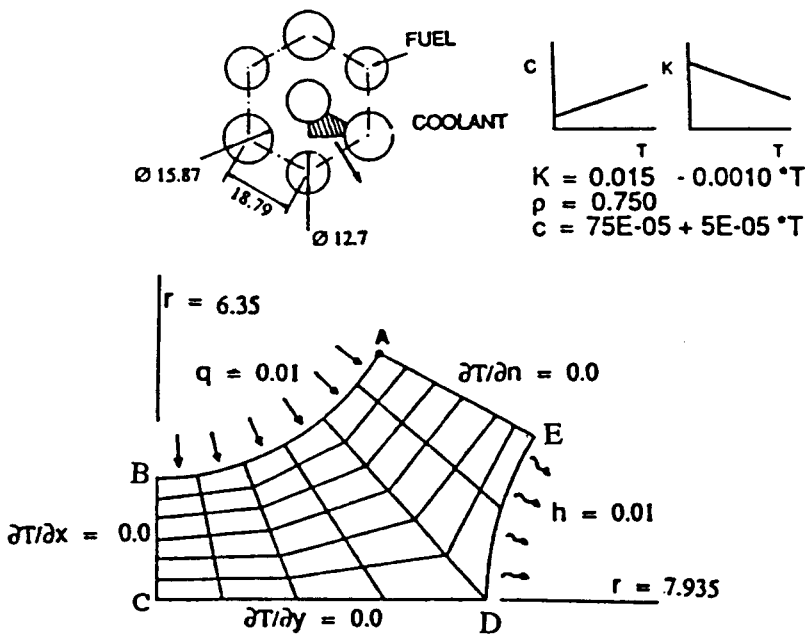


Figure 2. Description of a nuclear fuel element model and numerical data

ond-order accurate VIP explicit method of computation. The Cray XMP was used for the computations.

Test Case 1

A linear and non-linear multidimensional transient problem with constant and temperature-dependent properties is described here. The general configuration of a nuclear fuel element, the finite element mesh used, and relevant material and boundary condition data are given in Figure 2. A constant heat flux is assumed at $t = 0$ on edge AB. Edge DE is assumed to be convecting. The problem is modelled using two-dimensional bilinear elements.

Linear model. For the linear model, the comparative temperature histories (point A) and the corresponding errors are shown in Figures 3 and 4, respectively. The bench mark employed is the Crank–Nicolson with a very small time step $\Delta t = 0{\cdot}0025$ s. The improved stability characteristics of the VIP in comparison to the Crank–Nicolson method for a given $\Delta t = 1$ s is shown in

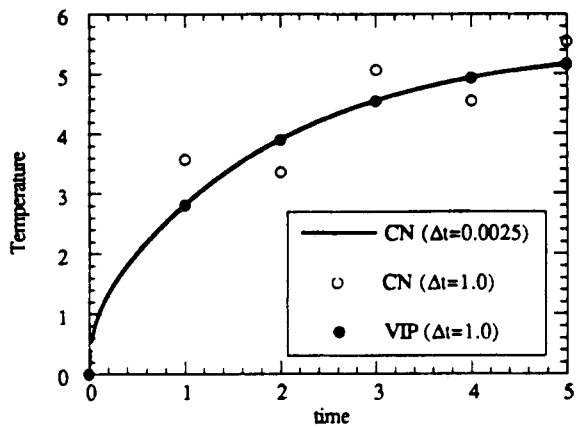


Figure 3. Comparative temperature histories for a given time step ($\Delta t = 1$ s): linear model

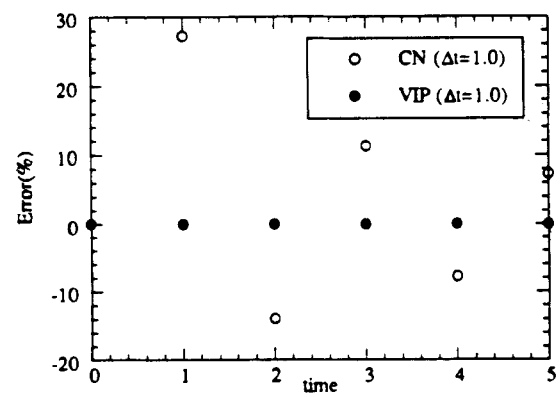


Figure 4. Comparative errors ($\Delta t = 1$ s): linear model Note: Even when $\Delta t = 5{\cdot}0$, VIP error is close to zero.

Figure 4. It should however be noted that for a given accuracy level (which is set to be 3 per cent), the Crank–Nicolson required a time step to be less than 0·005 s for which the CPU was 16·454 units in comparison to the VIP which needed a $\Delta t = 5$ s (CPU = 1·133). It should however be noted that for this linear case the VIP could have used a much larger time step and still be within the set accuracy level. In Figure 5 we also purposely show the VIP for a very small $\Delta t = 0{\cdot}01$ s to capture accurately the response in comparison to the implicit Crank–Nicolson which has a significantly larger error (approximately 11 per cent) at $\Delta t = 0{\cdot}01$ s.

Non-linear model. For the non-linear transient model, Figures 6 and 7 show the comparative temperature histories (point A) and errors for a given $\Delta t = 1$ s. The improved stability via the VIP is shown in Figure 7 for a $\Delta t = 1$ s. In Figure 8 we also show the improved accuracy for a small time step $\Delta t = 0{\cdot}02$ s. It should however be noted that for a given acceptable accuracy level (which is set to be 3 per cent), the Crank–Nicolson required a time step to be less than 0·005 s for which

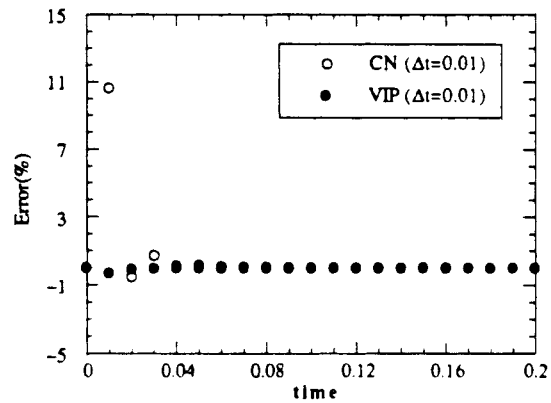


Figure 5. Comparative results for accuracy: linear model

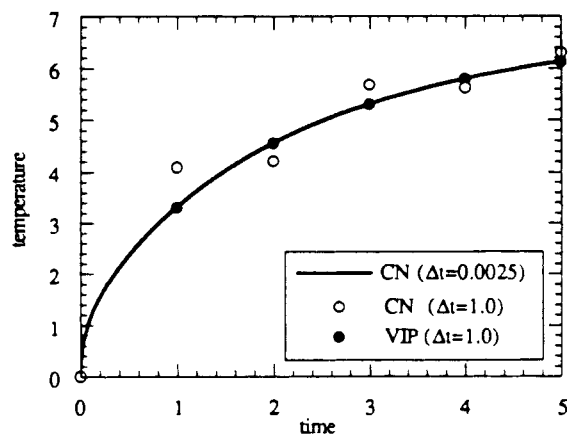


Figure 6. Comparative temperature histories for a given time step ($\Delta t = 1$ s): non-linear model

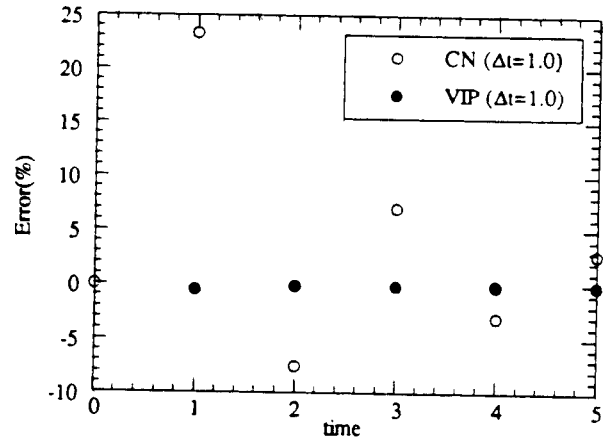


Figure 7. Comparative errors (Δt = 1 s): nonlinear model

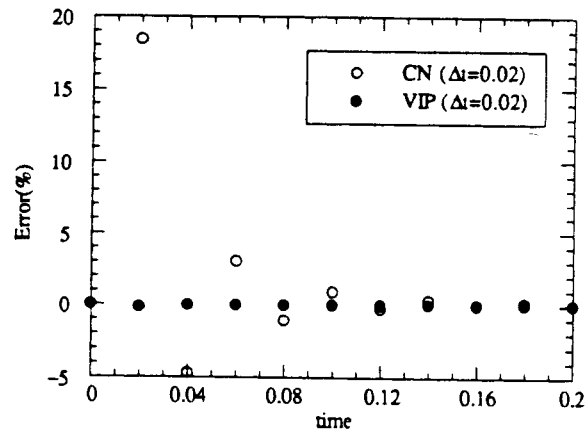


Figure 8. Comparative results for accuracy (Δt = 0·02 s): nonlinear model

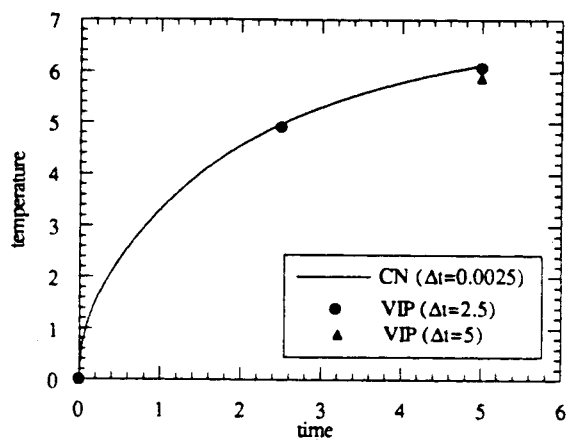


Figure 9. VIP method at various large time step: non-linear model

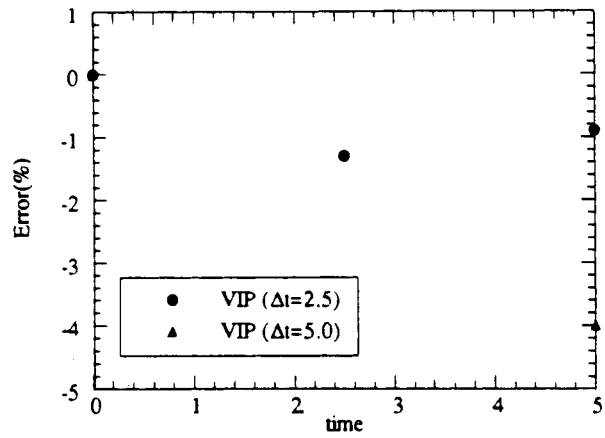
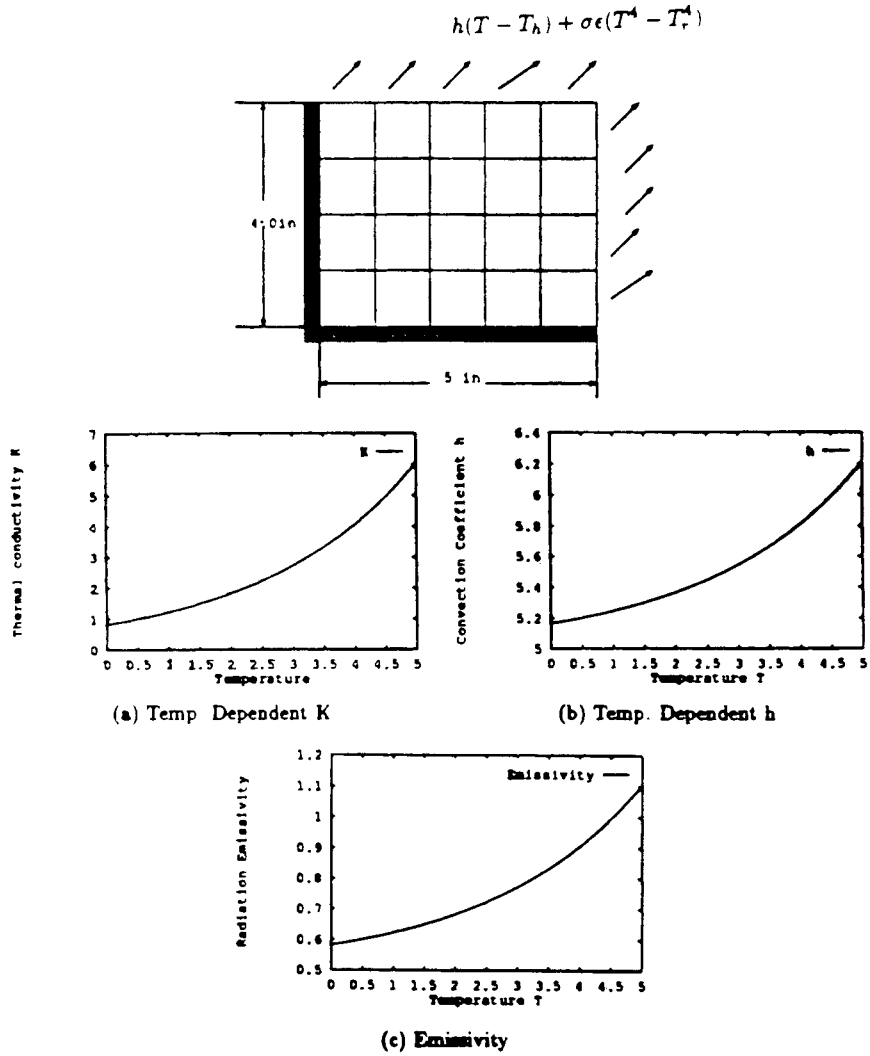


Figure 10. Accuracy of VIP method at various large time steps: non-linear model



(a) Temp Dependent K

(b) Temp. Dependent h

(c) Emissivity

Figure 11. Description of plate problem with edge convection and radiation

the CPU was 42·52 in comparison to the VIP which employed a $\Delta t = 2{\cdot}5$ s (for an error less than 3 per cent and CPU of 1·746) and $\Delta t = 5$ s (for error approximating 4 per cent and CPU of 1·585). Figures 9 and 10 show the plots of the VIP and the associated errors for $\Delta t = 2{\cdot}5$ and 5 s respectively.

The above examples demonstrated the applicability of the VIP method to a linear/non-linear multidimensional model. The improved accuracy/stability and effectiveness of the method are clearly evident.

Test Case 2

This test case involves a rectangular plate with non-linear boundary conditions (convection and radiation) and the thermophysical properties are temperature dependent. Only a quarter of the plate and a description of the problem is shown in Figure 11.

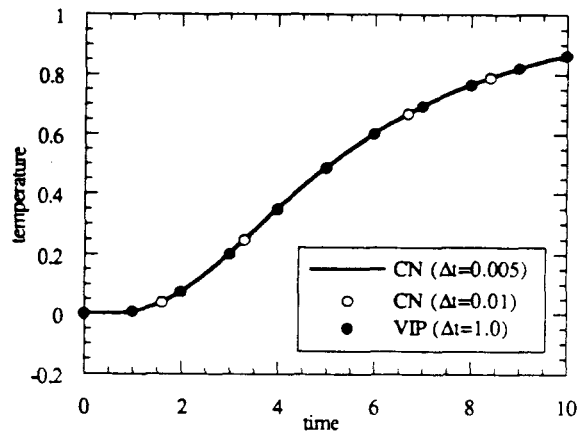


Figure 12. Comparative temperature histories showing superior time step advantage for a given accuracy 3 per cent

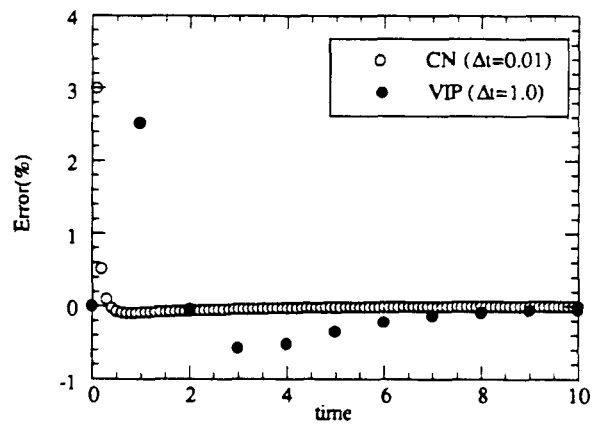


Figure 13. Comparative errors showing that for a given accuracy level 3 per cent, the VIP permits a large time step advantage

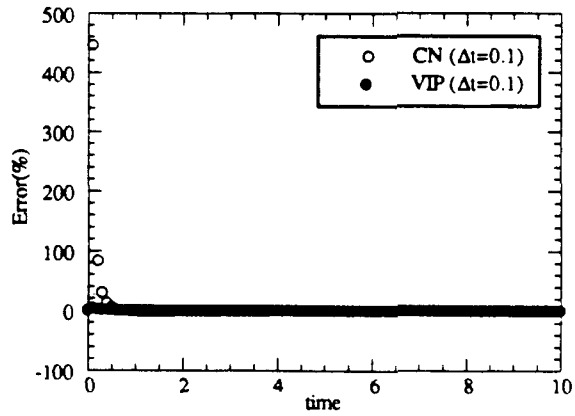


Figure 14. Comparative errors showing that the VIP method can accurately capture the early transient with negligible error

For a given accuracy level of 3 per cent, the Crank–Nicolson required a maximum time step of 0·01 s in comparison to the VIP which employed 1 s. The comparative temperature histories and the associated errors for the centre of the plate are shown in Figures 12 and 13, respectively. The comparative errors showing the accuracy of the VIP even at a small $\Delta t = 0{\cdot}1$ s to capture the early transient is shown in Figure 14. For assessing the effectiveness of the VIP method, a given accuracy level of 3 per cent was selected for which the comparative CPU times for the Crank–Nicolson versus the VIP method were 24·648 and 1·235 s, respectively. The overall effectiveness and accuracy of the VIP methodology is again evident from this multidimensional non-linear transient model.

CONCLUDING REMARKS

The paper outlined recent advances relevant to the theoretical and computational developments of a new Virtual-Pulse (VIP) explicit time integral methodology with emphasis on providing an effective generalized methodology of computation applicable to general transient thermal analysis of multidimensional non-linear/linear problems. Recent advances and progress towards providing an effective methodology with improved accuracy and stability characteristics were described via new and different perspectives and methodology of development. The numerical test models clearly validated the applicability of the VIP methodology to practical multidimensional thermal analysis situations. The choice of the number of modes selected may be problem dependent, and, the computation of the eigenproblem may play a role in certain cases; however, the VIP can be employed above a threshold time duration to offset the eigenproblem expense (part of this expense is offset by the larger time step permissible via the VIP methodology). When multiple load cases are involved, the VIP will indeed be an appropriate tool since the calculated eigenvalues/eigenvectors can be re-used for subsequent analyses. Unlike inertial problems in structural dynamics, where mode related approaches have indeed certain added advantages, typical thermal analysis problems belong more towards the propagation type and pose increased complexity. Nonetheless, the generalized explicit VIP time integral methodology described here is applicable to general linear/non-linear multidimensional thermal analysis problems. It should be clearly pointed out that although at a naive glance, the development of the methodology may

seem rather mathematically complex, in reality the explicit VIP methodology is fairly simple, straightforward and easy to implement. And, the approach possesses several computationally attractive and effective features which are also ideally suited for high-speed computing environments and for parallel computations, and for personal workstation computing environments.

ACKNOWLEDGEMENTS

The authors are very pleased to acknowledge support, in part, by NASA-Johnson Space Center/Lockheed Engineering and Space Science Co., Houston, TX. Related support, in part, and computer grants from the Army High Performance Computing Research Center (AHPCRC) at the University of Minnesota on a contract from the Army Research Office, and by the Minnesota Supercomputer Institute, Minneapolis, Minnesota are also greatly acknowledged.

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